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The Moore Penrose Inverse and Spectral Inverse of Fuzzy Matrices

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Abstract

In this paper we introduce the Moore Penrose inverse and Spectral inverse of fuzzy matrices using the structure of $M_n(F)$. The existence of group inverse and Drazin inverse of a square fuzzy matrices are discussed. Also the relation between group inverse and Moore Penrose inverse of range symmetric fuzzy matrices are studied. The existence of group inverse for the product of two square fuzzy matrices is investigated. The reverse order property for the Moore Penrose inverse of a fuzzy matrix is discussed.

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1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh in [8] 1965. Ragab. M. Z. and Emam E. G.[1] introduced the determinant and adjoint of a square fuzzy matrix. Zhang, K.L.[7] introduced the nilpotent matrices over D-lattices. Nagoorgani A. and KalyaniG.[4] discussed the Fuzzy m-norm matrices in the year 2003. Meenakshi A.R. and Inbam C. [3] introduced the concept of Drazin inverses of fuzzy matrices. In 2013, Nagoorgani A. and Manikandan A. R [5], introduced the concept On Fuzzy Det – Norm Matrix. Meenakshi A.R.[2] introduced the concept of Fuzzy matrix theory and its applications. Nagoorgani A. and Manikandan A. R [6] discussed the Finite Dimensional Fuzzy BI – Normed Linear Space in the year 2018.

In this paper, we first review some basic concepts of fuzzy matrices in section 2, in section 3 the definition of Moore Penrose Inverse of Fuzzy Matrices and the generalized inverse of a matrices are discussed. Also some of the spectral properties of the inverse of fuzzy matrices are verified. In section 4, the concepts of Spectral Generalized Inverse of fuzzy matrices are defined and some properties of Spectral and Moore Penrose Inverse of Fuzzy Matrices are discussed.

2. PRELIMINARIES

We consider $F = [0,1]$ the fuzzy algebra with operation [+,·] and the standard order "≤" where $a + b = max\{a, b\}$, $a \cdot b = min\{a, b\}$ for all a,b in F. F is a commutative semi-ring with additive and multiplicative identities 0 and 1 respectively. Let M_{MN}(F) denote the set of all $m \times n$ fuzzy matrices over F. In short M_n(F) is the set of all fuzzy matrices of order n. Define matrix addition and scalar multiplication in M_n(F) as A + B $= [a_{ij} + b_{ij}]$, where A = $[a_{ij}]$ and B = $[b_{ij}]$ and cA = [ca_{ij}], where c is in [0,1], with these operations M_n(F) forms a linear space.

3. SPECTRAL INVERSE OF FUZZY MATRICES

Definition 3.1

An $m \times n$ matrix $A = (a_{ij})$ whose components are in the unit interval [0,1] is called a fuzzy matrix.

Definition 3.2

For an fuzzy matrix A of order $m \times n$, an fuzzy matrix G of order $n \times m$ is said to be g inverse (or) ({1,2})-inverse or semi inverse of A. If AXA=A and XAX=X, where X is said to be g-inverse or a least square g-inverse of A. If AXA=A and $(AX)^T = AX$, where X is said to be g-inverse or a minimumα-norm g-inverse of A, If AXA=A and $(XA)^T = XA$, G is said to be a Moore-Penrose inverse of A in $M_n(F)$, if

$$AXA=A$$

$$XAX=X$$

$$(AX)^T = AX$$

$$(XA)^T = XA$$

and

$$A^k XA = A^k$$

$$AX=XA$$

$$A^k X = XA^k$$

$$AX^k = X^k A$$

The Moore-Penrose inverse of A is denoted by A^{+}.

Definition 3.3

The sequence fuzzy matrix A has a group inverse $A^{\#}$ if and only if $rank\ A^{k} = rank\ A^{k+1}$.

Definition 3.4

Let A in $M_n(F)$ be regular if and only if A has a g-inverse. If A is regular, then a

g- Inverse of A is denoted as A^{-} and A{1} is the set of all g-inverses of A satisfying $AA^{-}A = A$ for all A^{-} in A{1}.

Definition 3.5

If A in $M_n(F)$ is a range symmetric fuzzy matrix if and only if $R(A) = R(A^T)$

Let $R(M_n(F))$ be the set of all range symmetric fuzzy matrices in $M_n(F)$.

Theorem 3.1

A square fuzzy matrix A has a group inverse if and only if $rank\ A = rank\ A^2$.

Proof:

If A in $M_n(F)$ is nonsingular R(A) in $M_n(F)$ and N(A)={0}. Thus R(A) and N(A) are complementary in $M_n(F)$.

Since a nonsingular matrix has unique

For any k in [0,1]

$$dim\ R(A^k) + dim\ N(A^k) = rank\ (A^k) + null\ (A^k) = n$$

Since $R(A^k)$ and $N(A^k)$ are complementary in $M_n(F)$.

$$R(A^k) \cap N(A^k) = \{0\} \qquad (1)$$

Since for any k in [0,1]

$$R(A^{k+1}) \subset R(A^k)$$

$$N(A^k) \subset N(A^{k+1})$$

From(1)

$$dim\ R(A^k) = dim\ R(A^{k+1})$$

Theorem 3.2

If A is fuzzy matrix of $M_n(F)$.

(i) X exists

(ii)A^k is regular, $dim\ R(A)\ =\ dim\ R(A^k)$

(iii) Both the equations $A^kX = A$ and $XA^k = A$ can be solved for X

Proof:

(i)⇒(ii) If X exists

$$A = A^kX = XA^k$$

$$A = XA^k$$

$$dimR(A) = dimR(XA^k)$$

$$\leq dimR(A^k)$$

For a pair of fuzzy matrices X and A, if the product XA is defined then

$$dimR(XA) = dimR(X)\ \cdot A \subseteq dimR(A)$$

$$dimR(A) \subseteq dimR(A^k)$$

$$A = A^kX \Rightarrow dimR(A) = dimR(A^kX)$$

$$\subseteq R(A^k)$$

$$dimR(A) \subseteq dimR(A^k)$$

Therefore A is regular

(ii)⇒(iii) If A^k is regular

$$A^k\{1\} \neq \emptyset$$

$$R(A) = R(A^k)$$

$$A = A(A^k)^-A^k \text{ for all } (A^k)^- \text{ in } A^k\{1\}$$

For any $(A^k)^-$ the fuzzy matrix $(A^k)^-A$ is a solution for the fuzzy matrix equation

$$A^kX = A$$

and the fuzzy matrix $A(A^k)^-$ is a solution for the fuzzy matrix equation

$$XA^k = A$$

(iii) ⇒(i)

Let U and V be fuzzy matrix of$M_n(F)$.

$$A^kX = A = XA^k = A$$

For $Y = VAU$

$$AUA = (VA^k)UA$$

$$= V(A^kU)A$$
$$= VA^k$$
$$AUA = A \tag{2}$$
$$AVA = AV(A^kU)$$
$$= A(VA^k)U$$
$$= A^kU$$
$$AVA = A \tag{3}$$

Using (2) and (3)

$$\Longrightarrow AY = A(VAU)$$
$$= (AVA)U$$
$$= AU$$
$$= VA^kU$$
$$= VA$$
$$= V(AUA)$$
$$= (VAU)A$$
$$AY = YA$$
$$\Longrightarrow YAY = (VAU)A(VAU)$$
$$= V(AUA)VAU$$
$$= V(AVA)U$$
$$= VAU$$
$$YAY = Y$$
$$\Longrightarrow AYA = A(VAU)A$$
$$= (AVA)UA$$
$$= AUA$$
$$AYA = A$$

Hence Y=X is the group inverse of A in $M_n(F)$.

Theorem 3.3

If A is fuzzy matrix of $M_n(F)$, then A^+ exists. If A is range symmetric $\Longleftrightarrow A^{\#T}$ exists, $A^{\#T} = A^{T\#}$, where $A^\#$ is the group inverse of A.

Proof:

If A is range symmetric

$$R(A) = R(A^{T\#})$$

$$\Longrightarrow R(A) \subset \mathrm{R}(A^{T\#})$$

$$\Longrightarrow A = XA^{T\#}$$

$$= XA^{T\#}(A^{T\#})^{-}A^{T\#}$$

$A = A(A^{T\#})^{-}A^{T\#}$ for all $(A^{T\#})^{-}$ in $A^{T}\{1\}$

Since $(A^{T\#})^{+}$ is a g-inverse of $A^{T\#}$

$$A = A(A^{T\#})^{+}A^{T\#}$$

$$A = A(A^{T})^{T}A^{T\#}$$

$$= AAA^{T\#}$$

$$A = A^{2}A^{T\#}$$

$R(A^{T\#}) \subseteq \mathrm{R(A)} \Rightarrow A^{T\#}A^{-}A$ for all A^{-} in A{1}

$$=A^{T\#}(A^{+})^{T}A$$

$$=A^{T\#}A^{T\#}A$$

$$\mathrm{A}=(A^{T\#})^{2}A$$

$$\mathrm{A}A^{T\#} = AA^{+} = A((A^{T\#})^{2}A)$$

$$A^{+}A = A^{T\#}A = (A^{T\#}A^{2}(A^{T\#}$$

$$= A(A^{2})^{T\#}A$$

Since $A^{T\#}A$ is symmetric

$$=\mathrm{A}A^{T\#}$$

$$=\mathrm{A}A^{+}$$

Hence $\mathrm{A}A^{+} = A^{+}A$for all A^{+} in $A\{1,2\}$

$$\Longrightarrow A^{T\#} = A^{+} = A^{\#T}$$

Hence $A^{\#T}$ exists $A^{\#T} = A^{T\#}$

Conversely, if $A^{\#T}$ exists, then $A^{\#T} = (A^{\#T})^{2}A$

$$\Longrightarrow R(A^{\#T}) = R((A^{\#T})^{2}A) \subseteq R(A)$$

Since $A^{\#T} = A^{T\#} \Longrightarrow \mathrm{R}(A^{T\#}) \subseteq \mathrm{R(A)}$

$\mathrm{A}=A^{2}A^{\#T} \Longrightarrow \mathrm{R(A)} \subseteq R(A^{\#T})$

$$= \mathrm{R}(A^{T\#})$$

$\Longrightarrow$ R(A) $\subseteq R(A^{T\#})$

Hence A is range symmetric,$A^{\#T}$ in A{1,2} and $A^{\#T} = A^{T\#} \Longrightarrow A^{+} = A^{\#T}$

Hence A^{+} is exists.

Theorem 3.4

Let A in M_n(F), If $A^{\#}$ exists,

(i) $A^{\#}$ $=A^{\#}$AA=AA$A^{\#}$ (or) $A^{\#}$ $=A^{\#}A^{2}$=$A^{2}A^{\#}$

(ii) $A^{\#} = A(A^{3})^{(1)}A$, where $(A^{3})^{(1)}$ an element of $A^{3}\{1\}$

Proof:

(i)$R(A^{\#}) = R(A^{\#}A) \subseteq R(A) = R(AA)$

$\Longrightarrow$ R($A^{\#}$) $\subseteq$ R(AA)

$\Longrightarrow A^{\#} = A^{\#}(AA)^{-}AA$for all (AA) in A{1}

$\Longrightarrow A^{\#} =A^{\#}(AA)AA$

$= A^{\#}AA$

$A^{\#} = A^{\#}A^{2}$

$\Longrightarrow A^{\#} =$ AA $(AA)^{-}A^{\#}$

$= AA(AA)A^{\#}$

$=AAA^{\#}$

$A^{\#} = A^{2}A^{\#}$

(ii) Let X=$AA^{3}A$

Claim: X= $A^{\#}$

$\Longrightarrow A^{\#}XA^{\#} = A^{\#}(AA^{3}A)A^{\#}$

$=A^{\#}(AAAAA)A^{\#}$

$= (A^{\#}AA)A(AAA^{\#})$

$= (A^{\#}A^{2})A(A^{\#}A^{2})$

$= A^{\#}AA^{\#}$

$A^{\#}XA^{\#} = A^{\#}$

$\Longrightarrow XA^{\#}X = X(A^{\#}A^{\#}A^{\#}A^{\#}A^{\#})X$

$= (X(A^{\#})^{2})\ A^{\#}((A^{\#})^{2}X)$

$= XA^{\#}X$

$XA^{\#}X = X$

Also $A^{\#}X = A^{\#}(AAAAA)$

$= (A^{\#}AA)AAA$

$= A^{\#}AAA$

$A^{\#}X = A^{\#}A = XA$ is symmetric

Therefore $XA = AA \Rightarrow XA$ is symmetric.

Thus X in A{1}, Hence $A^{\#}$ exists and $A^{\#} = X = A(A^3)^{(1)}A$

4. SPECTRAL PROPERTIES OF THE DRAZIN INVERSE ON M_n(F)

Definition 4.1

For A in M_n(F) the Drazin inverse of denoted as A_d is a solution of the following equations,

$$AX = XA$$

$$X = X^2A$$

$$A^k = A^{k+1}X \text{For some k in [0, 1]}$$

The smallest positive integer k is called the index of A inM_n(F). Hence the group inverse is a particular case of Drazin inverse for fuzzy matrices in [0, 1].

Theorem 4.1

If G is a $\{1^l, 5\}$-inverse of square fuzzy matrix A in M_n(F). Then X=A^lG^{l+1} is a $\{1^l, 2{,}5\}$-inverse.

Proof:

If G is a g-inverse of square fuzzy matrix A in M_n(F)

$$A^{l+1}G = A^l$$

$$AG = GA$$

$$\Longrightarrow A^lXA = A^{2l+1}G^{l+1}$$

$$= A^{2l}G^l$$

$$=A^{2l-1}G^{l-1}$$

$$=A^l$$

$$\Longrightarrow XAX = A^{2l+1}G^{2l+2}$$

$$=A^{2l}G^{2l+1}$$

$$=A^lG^{l+1}$$

$$XAX = X$$

Hence $X = A^lG^{l+1}$ is a $\{1^l, 2{,}5\}$-inverse.

Definition4.2

A matrix A in M_n(F) is nilpotent if $A^p = 0$ for some p inN.

Theorem 4.2:

If A in M_n(F) has a decomposition of the form $A = B + N$ such that $B^\#$ exists. N is nilpotent and $BN = NB = 0$ then $B = (A_d)^\#$

Proof:

If $B^\# = B(B^\#)^2$

$= (B^\#)^2 B$

$\Longrightarrow B^\# N = NB^\# = 0$

Consequently,

$AB^\# = BB^\# = B^\# A$

$A(B^\#)^2 = B(B^\#)^2$

$= B^\#$

By using $BN = NB = 0$

$$A^l = (B + N)^l = B^l + N^l$$

If l is sufficiently large so that$N^l = 0$, $A^l = B^l$ and for such l

$\Longrightarrow A^{l+1}B^\# = B^{l+1}B^\# = B^l$

$\Longrightarrow A^{l+1}B^\# = A^l$

$A(B^\#)^2 = B^\#$and$AB^\# = B^\# A$

$\Longrightarrow B^\# = A_d$

$B = (A_d)^\#$

The relation between the Drazin inverses of AB and BA in M_n(F),

Therefore the decomposition is unique if it is exists.

Theorem 4.3

Let A be fuzzy matrix inM_n(F), then A and X are g-inverse of each other if $X_d = (A_d)^\#$ and if X in $A\{1\} \cup A\{2\}$. It is an g-inverse of A in M_n(F).

Proof:

If A and X have the same range and the same null space.

$$XX_d \text{ and } AA_d = A_d(A_d)^\#$$

It l is the maximum of the indices of A in $M_n(\mathrm{F})$ and X is the fuzzy matrix of $M_n(\mathrm{F})$.

$$\Longrightarrow XA^{l+1} = X(A_d)^{\#}A_dA^{l+1}$$
$$= XX_dA^l$$
$$XA^{l+1} = A^l$$
$$\Longrightarrow AX^{l+1} = A(X_d)^{\#}X_dX^{l+1}$$
$$= AA_dX^l$$
$$XA^{l+1} = X^l$$

SinceA_d andX_d the same null space. Let A and X be g-inverses in $M_n(\mathrm{F})$ and X in A{1}.

$$N(A_d) = N(X_d)$$
$$(A_d)^{\#}X^N = (X_d)^{\#}A^N = 0$$
$$R(A_d) = R(X_d)$$
$$N(A_d^*) = N(X_d^*)$$
$$X^N(A_d)^{\#} = (X_d)^{\#}A^N = 0$$

Consequently,

$$A = AXA = (A_d)^{\#}(X_d)^{\#}(A_d)^{\#} + A^NX^NA^N$$
$$A_d = A_dAA_d$$
$$= AA_d(X_d)^{\#}AA_d$$
$$A_d = (X_d)^{\#}$$

Since AA_d is the projector on the range of $(X_d)^{\#}$ along its null space.

Hence X in $A\{2\}$, it is an g-inverse of A in $M_n(\mathrm{F})$.

Definition: 4.3

For A in $M_n(\mathrm{F})$, A^{i*} denotes the i[th] row of A and A^{*j} denotes the j[th] column of A.

Theorem 4.4

For some pair of fuzzy matrices A, B in $M_n(\mathrm{F})$, then $(AB)^+ = B^+A^+$ if and only if $R(A^*AB) \subset R(B)$ and $R(BB^*A^*) \subset R(A^*)$.

Proof:

If $$BB^+A^*AB = A^*AB$$
$$B^*AABB^+ = B^*A^*A$$

The multiplying on the right by A^+ and on the left by $(AB)^{*+}$

$$ABB^+A^+ = AB(AB)^+$$

$$B^+A^+ = (AB)^+ \quad (4)$$

and

$$A^+ABB^*A^* = BB^*A^*$$

The multiplying on the left by B^+ and on the left by $(AB)^{*+}$,

$$B^+A^+\text{AB} = (AB)^+AB$$

$$B^+A^+ = (AB)^+ \quad (5)$$

From (4) and (5) that B^+A^+ in $(AB)\{1,3,4\}$

$$\Longrightarrow B^*A^* = B^*BB^+A^+AA^*$$

$$B^+A^+ = B^+B^{*+}B^*A^*A^{*+}A^+$$

$$B^+A^+ = B^*A^*$$

$$rank\ (B^+A^+) = rank(B^*A^*)$$

$$= rank(AB)$$

Therefore B^+A^+ in $(AB)\{2\}$

$$\Leftarrow B^*A^* = B^+A^+ABB^*A^*$$

Multiplying on the left by ABB^*B gives,

$$ABB^*BB^*A = ABB^*A^+ABB^*A^*$$

Since the left member is Hermitian and $\text{I} - A^+A$ is idempotent.

$$BB^*A^* = A^+ABB^*A^*$$

$$BB^*A^* = A^*$$

$$R(BB^*A^*) \subset R(A^*)$$

Since

$$BB^+A^*AB = A^*AB$$

$$A^*AB = B$$

$$R(A^*AB) \subset R(B)$$

This is the reverse order property for the Moore Penrose inverse.

Definition: 4.4

For A in $M_n(\text{F})$ if there exists X in $M_n(\text{F})$ such that $AXA = A$ and $XAX = X$ then X is called a semi- inverse (or) {1,2} inverse of A, set of {1,2} inverse of A is denoted as $A\{1,2\}$.

Theorem: 4.5

Let A in M_n(F) then A can be written as $A = GE = EH$, Where E in M_{mn}(F) is a partial isometric and G in M_{mm}(F), H in M_{nn}(F) are hermitian and positive semi-definite. The fuzzy matrices E, G and H are uniquely determined by

$$R(E) = R(G)$$

$$R(E^*) = R(H)$$

$$G^2 = AA^*$$

$$H^2 = A^*A$$

and E is given by

$$E = U_r V_r^*$$

Proof:

Let A=UDV^*,D= $\begin{bmatrix} \alpha_1 & : & 0 \\ & : & \\ \cdots\cdots & \alpha_r \cdots & \cdots \\ 0 & : & 0 \end{bmatrix}$ be the singular-value decomposition of Ain M_n(F). For any k, $r \le k \le \min\{m, n\}$, define the three fuzzy matrices D_k= $\begin{bmatrix} \alpha_1 & \\ & \ddots \\ \alpha_r & \\ & . \end{bmatrix}$

$$U_{(k)} = [u_1 \cdots u_k] \in C^{m\times k},\ V_{(k)} = [v_1 \cdots v_k] \in C^{n\times k}$$

A=$U_{(k)} D_{(k)} V_{(k)}^*$

= $(U_{(k)} D_{(k)} U_{(k)}^*)(U_{(k)} V_{(k)}^*)$, since $U_{(k)}^* U_{(k)}$=I_k

= $(U_{(k)} V_{(k)}^*)\ (V_{(k)} D_{(k)} V_{(k)}^*)$, since $V_{(k)}^* V_{(k)}$=I_k

If $A = GE = EH$with the partial isometric.

$$E = U_k V_k{}^*$$

the positive semi definite fuzzy matrices.

$$G = U_k D_k U^*{}_k,\ H = V_k D_k V^*{}_k$$

Let E be non unique if r < min{m,n}, and G and H are also non unique,

$$G = U_k D_k U^*{}_k + U_{k+1} U_{k+1}{}^*$$

$$H = V_k D_k V^*{}_k + V_{k+1} V_{k+1}{}^*$$

The satisfies $A = GE = EH$ for the E given in $E = U_k V^*{}_k$.

Let E and G satisfy $R(E) = R(G)$ then from $A = GE = EH$

$$AA^* = GEE^*G = GEE^+G = GP_{R(E)}G = G^2$$

If $G^2 = AA^*$, The uniqueness of E follows from,

$$E = EE^+E = GG^+E = G^+GE = G^+A$$

$$\Rightarrow R(E^*) = R(H) \Leftrightarrow H^2 = A^*A$$

Let E^* and H satisfy $R(E^*) = R(H)$ in M_n(F).

$\Rightarrow A = GE^* = E^*$ H

$A^*A = E^*HEH$

$= HEE^*H$

$= HEE^+H$

$= HP_{R(E)}H = H^2$

$H^2 = A^*A$

The uniqueness of E follows from,

$E^* = EE^*E$

$= EE^+E$

$= HH^+E$

$= H^+EH = H^+A$

Hence the uniqueness of H, E in M_n(F).

$\Rightarrow G^2 = AA^*$

$= U_rD_rV^*{}_r V_rD_rU_r$

$= U_rD^2{}_rU^*{}_r$

$G = U_rD_rU^*{}_r$

Consequently,

$\Rightarrow G^+ = U_rD^{-1}{}_rU^*{}_r$

$E = G^+A$

$= U_rD^{-1}{}_rU^*{}_r U_rD_rV^*{}_r$

$E = D_rV^*{}_r$

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