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Some Inequality of Fuzzy Matrices

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Abstract: *In this paper, the fuzzy matrix geometric mean is fuzzy concave. We complete this important fact with a reverse result. This follows from an interesting non-commutative extension of a classical reverse Cauchy Schwarz inequality in* $M_n(F)$.
Keywords: *Fuzzy matrix, determinant of a squarefuzzy matrix, ,, fuzzy norm, Cauchy Schwarz inequality*

I. Introduction

The concept of fuzzy set was introduced by Zadeh [10] in 1965. R. Biswas [2] and A. M. K.H.kim and F.W.Roush [4] first tried to give a meaningful definition of Generalized fuzzy matrices and associated fuzzy norm function. Jian Miao Chen[3] introduced Fuzzy matrix partial ordering and generalized inverse. [1]Bertoiuzzaintroduced the properties ofdistributivity of t-norm and t-conorms.[5] Ragab.M. Z. and Emam E. G. introduced the concept of determinant and have the developed adjoint of a square fuzzy matrix. introduced On fuzzy 2-normed linear spaces. Meenakshi andCokilavany [6],[7,8] introduced the properties of fuzzy m-norm matrices and Binormed sequences in fuzzy matricesGani and Kalyani. Introduced the definition of positive Definite Matrices[2] R.Bhatia. Recently [9] ZHOU Min - na have introduced a definition of Characterizations of the Minus Ordering in Fuzzy Matrix Set` In this paper, the concept of inequalityof fuzzy matrices on M_n(F), the set of all fuzzy sets of $P^*(M_n(\mathrm{F}))$, the standard Cauchy Schwarz inequality is proved. Moreover Cauchy Schwarz inequality of fuzzy matrices are established. Some equivalent conditions are also proved.

II. Preliminaries

We consider F=[0,1] the fuzzy algebra with operation [+,·] and the standard order "≤" where a+b = max{a,b}, a·b = min{a,b} for all a,b in F.F is a commutative semi-ring with additive and multiplicative identities 0 and 1 respectively. Let M_{MN}(F) denote the set of all m×n fuzzy matrices over F. In short M_n(F) is the set of all fuzzy matrices of order n. define '+' and scalar multiplication in M_n(F) as $A + B = [a_{ij} + b_{ij}]$, where A = $[a_{ij}]$ and B = $[b_{ij}]$ and cA = $[ca_{ij}]$, where c is in [0,1], with these operations M_n(F) forms a linear space.

Definition 2.1
An m×n matrix $A = [a_{ij}]$ whose components are in the unit interval [0,1] is called a fuzzy matrix.
Definition 2.2
The determinant |A| of an n × n fuzzy matrix A is defined as follows;

$$|A| = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Where S_n denotes the symmetric group of all permutations of the indices $(1,2,\cdots,n)$
Definition 2.3
A fuzzy matrix A is defined to be greater than B if $\|B\| \le \|A\|$, A is strictly greater than B if $\|B\| < \|A\|$. We also say that B is smaller than A.
Definition 2.4
Define a mapping $d{:}M_n(F) \times M_n(F) \to [0,1]$ as$\|A + B\| = \det[A, B]$ for all A,B in $M_n(F)$.
Definition 2.5
A matrix A in M_n(F) is called idempotentif $A^2 = A$
or $\|A^2\| = det[A]$, where $A = [a_{ij}]$

Definition 2.6

Let A be in $M_n(F)$ and α be in [0,1] such that ||A||= α or $n(P_A) = \alpha$, then the pair (P_A, α) is called a fuzzy point in $M_n(F)$ and it is denoted by P_A^α. The dual fuzzy point P_A^α is the point with norm (1- α) denoted by $P^* = P^{1-\alpha}$.

Definition 2.7

The set of all fuzzy points in $M_n(F)$ is gven by

$$P^*\big(M_n(F)\big) = \{P_A^\alpha |\ A \epsilon\ M_n(F),\quad \alpha\ \epsilon\ [0,1]\}$$

In F we follow the usual ≤ order relation correspondingly we define an order relation in $P^*\big(M_n(F)\big)$

Definition 2.8

The Geometric mean $\sqrt{ab}$ of two positive real numbers a,b$(0 \le a, b\ \le 1)$ is a convex operation. This paper equivalent to the Cauchy-Schwarz inequality. Let $P_A, P_B, \dots, P_Z$ be n × n Fuzzy Matrices or operators on a n-dimensional space $P^*\big(M_n(F)\big)$, for $P_A, P_B\ \in\ M_n(F)$ their geometric mean $P_A \# P_B$ is defined by two quite natural requirements.

1. $P_A P_B\ =\ P_B P_A$ implies $P_A \# P_B = \sqrt{P_A P_B}$
2. $(P_X^* P_A P_X)\ \#\ (P_X^* P_B P_X)\ =\ P_X^*(P_A \# P_B) P_X$

for any invertible P_X on $M_n(F)$.

Then, $P_A \# P_B\ =\ P_A^{1/2}\left(I\ \#\ P_A^{-1/2} P_B P_A^{-1/2}\right) P_A^{1/2}$

$$P_A \# P_B\ =\ P_A^{1/2}\left(P_A^{-1/2} P_B P_A^{-1/2}\right) P_A^{1/2} \qquad \text{-----------(1)}$$

So that $P_A \# P_B$ should be solution of $P_Z\, P_A^{-1} P_Z\ =\ P_B$, $P_Z\ in\ M_n(F)$or equivalently to $P_Z\, P_B^{-1} P_Z\ =\ P_A$.

Hence, $P_A \# P_B$ can be defined (1) and $P_A \# P_B\ =\ P_B \# P_A$.

Since $P_A \# P_B$is operator increasing. Since f(t) = $t^{1/2}\ is$ operator monotone.

Theorem 2.9

Let $P_A, P_B > 0\ in\ M_n(F)$. Then $P_A \# P_B\ = Max\ \left\{P_X\ \in [0,1]/\begin{pmatrix} P_A & P_X \\ P_X & P_B \end{pmatrix} \ge 0\right\}$.

Proof:

$\begin{pmatrix} P_A & P_X \\ P_X & P_B \end{pmatrix} \ge 0$, if P_A, P_B, P_X in $M_n(F)$ where $P_X\ =\ P_A^{1/2}\ K\ P_B^{1/2}$ for some contraction K ∈ [0,1] and n(K) ≤ 1.

$$\Rightarrow n\ \left(P_A^{-1/2} P_X P_B^{-1/2}\right)\ \le 1$$

(or)

$$n\ \left(P_A^{-1/2} P_X P_A^{-1/2} P_A^{1/2}\ \ P_B^{-1/2}\right)\ \le 1$$

(or)

$$\left(P_A^{-1/2} P_X P_A^{-1/2}\right)^2\ \le\ P_A^{-1/2} P_B P_A^{-1/2}$$

Therefore, by operator monotony of t → $t^{1/2}$.

$$P_A^{-1/2} P_X P_A^{-1/2}\ \le\ \left(P_A^{-1/2} P_B P_A^{-1/2}\right)^{1/2}$$

Hence $P_X\ \le\ P_A^{1/2}\left(P_A^{-1/2} P_B P_A^{-1/2}\right)^{1/2} P_A^{1/2}\ =\ P_A \# P_B$

Next we have check that,

$$\begin{pmatrix} P_A & P_A \# P_B \\ P_A \# P_B & P_B \end{pmatrix}\ \ge 0$$

$\Rightarrow n\left(P_A^{-1/2}(P_A \# P_B) P_B^{-1/2}\right) = 1$since$P_A^{-1/2}(P_A \# P_B) P_B^{-1/2}$is unitary.

Theorem 2.10

Let $P_A, P_B \in M_n(F)$such that $\alpha P_A \geq P_B \geq \beta P_A$ for some $0 \leq \alpha, P \leq 1$ and let Φ be a positive linear map then $\Phi(P_A) \# \Phi(P_B) \leq \frac{(\alpha/\beta)^{1/4} + (\beta/\alpha)^{1/4}}{2} \Phi(P_A \# P_B)$.

Proof:

Step 1

Suppose that for a vector f,

$$\Phi(P_A) = (f, P_A f)$$

$$P_Z = \left(P_A^{-1/2} P_B P_A^{-1/2}\right)^{1/2} \quad and \quad h = P_A^{1/2} f$$

Since $\alpha P_A \geq P_B \geq \beta P_A$ implies

$$\alpha^{1/2} \geq \left(P_A^{-1/2} P_B P_A^{-1/2}\right)^{-1/2} \geq \beta^{1/2}$$

$$\{\langle f, P_A f\rangle, \langle f, P_B f\rangle\}^{1/2} \leq \frac{(\alpha/\beta)^{1/4} + (\beta/\alpha)^{1/4}}{2} (f, P_A \# P_B f).$$

Step 2

Let h be any vector. Then by Fact 1

$$\langle h, \Phi(P_A) \# \Phi(P_B)\, h\rangle \leq \langle h, \Phi(P_A) h\rangle^{1/2} \langle h, \Phi(P_B) h\rangle^{1/2}$$

$$= \Psi(P_A)^{1/2} \Psi(P_B)^{1/2}$$

Where Ψ is defined by $\Psi(P_X) = \langle h, \Psi(P_X) h\rangle$

By fact 2: $\Psi(P_X) = T_r P_Y P_X = \langle P_X^{1/2}, \prod(P_X) P_Y^{1/2}\rangle$ for some $P_Y \in M_n(F)$

Since $\alpha P_A \geq P_B \geq \beta P_A$ implies $\alpha \prod(P_A) \geq \prod(P_B) \geq \beta \prod(P_A)$

$$\Psi(P_A)^{1/2} \Psi(P_B)^{1/2} \leq \frac{(\alpha/\beta)^{1/4} + (\beta/\alpha)^{1/4}}{2} \Psi(P_A \# P_B).$$

Combining with the previous inequality, we get

$$\langle h, \Phi(P_A) \# \Phi(P_B)\, h\rangle \leq \frac{(\alpha/\beta)^{1/4} + (\beta/\alpha)^{1/4}}{2} \langle h, \Phi(P_A) \# \Phi(P_B)\, h\rangle$$

Theorem 2.11

Let P_Z in $M_n(F)$ with extremal eigen values a, b for all vectors h in $M_n(F)$.

$n(h) n(P_Z h) \leq \frac{(a/b)^{1/2} + (b/a)^{1/2}}{2} \langle h, P_Z h\rangle$.

Proof:

Fact 1: For P_A, P_B in $M_n(F)$ all vectors h and all positive linear maps Φ.

$\langle h, \Phi(\mathrm{P_A} \# P_B)\, h\rangle \leq \langle h, \Phi(P_A) h\rangle^{1/2} \langle h, \Phi(P_B) h\rangle^{1/2}$ -----(2)

$$\langle h, \mathrm{P_A} \# P_B\, h\rangle \leq \langle h, P_A h\rangle^{1/2} \langle h, P_B h\rangle^{1/2}$$

Inequality (2),

$$\Phi(\mathrm{P_A} \# P_B) \leq \Phi(P_A) \# \Phi(P_B)$$

Where $\Phi(P_A) = P_Z^* \mathrm{P_A} \mathrm{P_Z}$ for some n$\times$ k $fuzzy$ $matrix$ $\mathrm{P_Z}$.

$$\langle h, \mathrm{P_A} \# P_B\, h\rangle = \langle P_A^{1/2} h, (P_A^{-1/2} P_B P_A^{-1/2}) P_A^{1/2} h\rangle$$

$$\leq n\left(P_A^{1/2} h\right) n(\left(P_A^{-1/2} P_B P_A^{-1/2}\right) P_A^{1/2} h)$$

$$\langle h, \mathrm{P_A} \# P_B\, h\rangle \leq \langle h, P_A h\rangle^{1/2} \langle h, P_B h\rangle^{1/2}$$

Fact 2: let Φ be a positive fuzzy linear function on $P^*(M_n(F))$. Then there exists $\mathrm{P_X}$ in $M_n(F)$ such that $\Phi(P_A) = T_r P_A P_X$.

Hence if $\prod(P_A): P^*(M_n(F)) \times P^*(M_n(F)) \rightarrow [0,1]$ is the left multiplication by P_A.

$$\Phi(P_A) = \langle h, \prod(P_A) h\rangle$$

Where the fuzzy inner product is the canonical inner product on $M_n(F)$and $h = P_X^{1/2}$.

Theorem 2.12

Let $P_A, P_B \;\; in \; M_n(F)$ such that $\alpha \, P_B(u) \supset \; P_A(u) \; \supset \; \beta \, P_B(u)$ for some $\alpha, \beta \; \in [0,1]$. Then $\sum \lambda_j \, (P_A) \lambda_j(P_B) \;\; \leq \frac{\alpha+\beta}{2\sqrt{\alpha\beta}} T_r P_A P_B$.

Proof:

We may assume $P_A, P_B \;\; in \; M_n(F)$ and the inclusion conditions on $P_A(u), P_B(u)$ of $P^*\big(M_n(F)\big)$.

$$\Rightarrow \; \alpha \; \geq \; |P_B^{-1} P_A| \; \geq \; \beta \quad equivalently \; \alpha \; \geq \; |P_A P_B^{-1}| \; \geq \; \beta$$

$$\therefore n(P_A h) \, n(P_B h) \quad \leq \quad \frac{\alpha+\beta}{2\sqrt{\alpha\beta}} \langle h, P_A^2 \# \, P_B^2 \;\, h \rangle - - - - (3)$$

That is, $\Phi(P_X) \; = \; \langle h, P_X h \rangle$, denoted by $\|.\|_1 \; or \; n_1$ the trace norm. There exists a unitary V such that $\sum \lambda_j \, (P_A) \lambda_j(P_B) \; = \; n_1 \, (P_A \; \vee \; P_B)$

Where V = $\sum v_i h_i \otimes h_i$, Hence making use of the triangle inequality for the trace norm.

$$\sum \lambda_j \, (P_A) \lambda_j(P_B) \; \leq \; \sum n(P_A \, h_i) \;\; n(P_B h_i)$$

Combining with (2), we get

$$\sum \lambda_j \, (P_A) \lambda_j(P_B) \; \leq \quad \frac{\alpha+\beta}{2\sqrt{\alpha\beta}} T_r P_A^2 \# \, P_B^2$$

Next to check that,

$T_r P_A^2 \# \, P_B^2 \; \leq \; T_r P_A P_B$.

there is a unitary ∪ such that $P_A^2 \# \, P_B^2 \;\; = \;\; P_A \; \cup \; P_B$

$$\Rightarrow \; T_r P_A^2 \# \, P_B^2 \;\; = \;\; T_r P_A \; \cup \; P_B$$

$$= T_r \left(P_A^{1/2} \; \cup \; P_B^{1/2} \right) \, (P_B^{1/2} P_A^{1/2})$$

$$\leq \quad (T_r P_A \; \cup \; P_B \cup^*)^{1/2} (T_r P_A P_B)^{1/2}$$

$$\leq \quad (T_r P_A \; \cup \; P_B)^{1/2} (T_r P_A P_B)^{1/2}$$

$$T_r P_A \; \cup \; P_B \quad \leq \quad T_r P_A P_B$$

III. Conclusion

In this paper, the inequality of fuzzy matricesand it'sSome Cauchy Schwarz inequality in $M_n(F)$ are discussed. Numerical examplesare given to clarify the developed theory and the proposed inequalityof fuzzy matrices.

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