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Article

Self-Adjoint Fuzzy Linear Operator in a Fuzzy Hilbert Space

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ABSTRACT: The adjoint fuzzy linear operator and the self-adjoint fuzzy linear operator working on a fuzzy Hilbert space (FH-Space) are the subjects of this work-study. The features of the adjoint and self-adjoint fuzzy operators in an FH-adjoint fuzzy operators in an FH-space are discussed in detail along with a number of definitions, several elementary propositions on positive fuzzy operators, and numerous theorems.
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INTRODUCTION

In 1965, Zadeh [7] developed the idea of a fuzzy set. In addition to proving a fixed point theorem for fuzzy mappings, Heilpern [3] proposed the idea of fuzzy mappings as a mapping from an arbitrary set to one subset of fuzzy sets in a metric linear space. The result of Heilpern was expanded and broadened by several authors[5], [6]. We demonstrate fixed point theorems for fuzzy mappings that Heilpern proposed and applied to Hilbert spaces in the current study [3]. The idea of fuzzy inner product space (FIP-space), proposed by Felbin [1], Gani and Manikandan [4], can be seen as a generalisation of the idea of inner product space. Fuzzy Hilbert space and its applications were studied by Goudarzi, Mand Vaezpour, and S. M. [2] in 2009.

The following rules apply to the paper: There are various early findings in Section 2. The concept of adjoint fuzzy linear operators, self-adjoint fuzzy linear operators, many theorems, and a discussion of some of these fuzzy operators' features are introduced in section three.

PRELIMINARIES

In the following discussions, we mainly follow the definitions and notations due to Heilpern

Let H be a Hilbert space and F(H) be collection of all fuzzy sets in H. Let$P_A \in F(H)$ and $\alpha \in [0,1]$.Theα-level set of A, denoted by A_α is defined as

$$A_\alpha = \{x: A(x) \geq \alpha\} \text{if} \alpha \in (0,1]$$
$$A_0 = \{x: A(x) > 0\},$$

Where $\overline{B}$ stands for the closure of a set B.

Definition

A fuzzy subset A of F(H) is said to be an approximate quantity iff its α-level set is a nonfuzzy compact convex subset of F(H)for each $\alpha \in [0, 1]$ and $\sup_{x \in F(H)} A(x) = 1$. From the collection F(H), the sub collection of all approximate quantities is denoted by W(H).

Definition

Let A be in F(H) and α be in [0,1] such that $\|A\|=\alpha$ or $n(P_A) = \alpha$, then the pair (P_A, α) is called a fuzzy point in F(H) and it is denoted by P_A^α. The dual fuzzy point P_A^α is the point with norm (1-α) denoted by $P^* = P^{1-\alpha}$.

Definition

The set of all fuzzy points in F(H) is gven by $P^*(F(H)) = \{P_A^\alpha | A \epsilon F(H), \quad \alpha \epsilon [0,1]\}$. In F we follow the usual $\leq$ order relation correspondingly we define an order relation in $P^*(F(H))$.

Definition

We define $P_A^\alpha < P_B^\beta$ iff $\alpha < \beta$ and $P_A^\alpha = P_B^\beta$ iff$A = B$ (then automatically $\alpha = \beta$).

Definition

A fuzzy Hilbert space $(F(H))$is a vector space over $[0,1]$with a mapping $P^*(F(H)) \times P^*(F(H)) \rightarrow [0,1]$called the scalar product and denoted by(P_x, P_y)which satisfies the following conditions

i) $(P_x, P_y) = (P_y, P_x)$
ii) $(P_{x_1} + P_{x_2}, P_y) = (P_{x_1}, P_y) + (P_{x_2}, P_y)(P_{x_1}, P_{x_2}, P_y \in P^*(F(H)))$
iii) $(\alpha P_x, P_y) = \alpha(P_x, P_y)(P_x, P_y \in P^*(F(H)), \alpha \in [0,1])$
iv) $(P_x, P_y) > 0 for x \neq 0; (P_x, P_x) = 0 for P_x = 0(P_x \in P^*(F(H)))$
v) $P^*(F(H))$ is a Banach space with the norm $n(P_x) = (P_x, P_x)^{\frac{1}{2}}$.

Let $F(H)$ be are al or complex Hilbert space, and let S denote the class of all F - bounded symmetric operators in $F(H)$ and F- bounded linear mappings of T into it self such that

$$(TP_x, P_y) = (P_x, TP_y)(P_x, P_y \in P^*(F(H)))$$

Where S is the class bounded self adjoint operators, $S = S^*$. A relation $\leq$ is introduced into S by writing $A \leq B$ or $B \geq A$ to denote that

$$(P_A P_x, P_x) \leq (P_B P_x, P_x)(P_x \in P^*(F(H))).$$

The operators T belonging to S that satisfy T≥0 are called positive operators.

Definition

Let $(E, F, *)$be a probabilistic inner product space.

1. Asequence$\{P_{x_n}\} \in E$ is called F–converges to $P_x \in E$,If for any $\epsilon > 0$ and $\lambda > 0, \exists N \in Z+, N = N(\epsilon, \lambda)$ Such that $Fx_{n-x}, x_{n-x}(\epsilon) > 1 - \lambda$ when ever $n > N$.
2. A linear functional$f(P_x)$defined on E is called $F-$continuous, if $P_{x_n} \rightarrow P_x$ implies $f(P_{x_n}) \rightarrow f(P_x)$ for any $\{P_{x_n}\}, P_x \in E$.

Definition

Let $(E, G, *)$ be a F(H)- space with IP:$(u, v) = sup\{x \in \mathbb{R}: G(u, v, x) < 1\}, \forall u, v \in E$ and let $S \in FB(E)$, then S is self–adjoint Fuzzy operator, if$S = S^*$where S^* is adjoint Fuzzy operator of S.

SOME ELEMENTARY PROPOSITIONS ON POSITIVE FUZZY OPERATORS

Proposition. 1

For any positive operator T, the generalized Schwartz inequality holds

$$|(TP_x, P_y)|^2 \leq (TP_x, P_x)(TP_y, P_y)$$

Proof.

If $B(P_x, P_y) = (TP_x, P_y)$is a positive semi-definite symmetric bi-linear form and So the generalized Schwarz inequality for this form.

Proposition. 2

If T is a positive operator, then $n(T) = sup\{(TP_x, P_x): n(P_x) \leq 1\}$

Proof.

Let T be a positive operator and let $M = sup\{(TP_x, P_x) : n(P_x) \leq 1\}$

By the Schwarz inequality

$$|(TP_x, P_x)| \leq n(TP_x)n(P_x),$$

$$M \leq n(T) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

putting P_y=TP_x in the generalized Schwarz inequality, we have

$$n(TP_x)^4 = (TP_x, TP_x)^2 = |TP_x, P_y|^2 \leq (TP_x, P_x)(TP_y, P_y)$$
$$\leq M^2 n(P_x)^2 n(TP_x)^2,$$
$$n(T) \leq M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2)

$n(T) = sup\{(TP_x, P_x): n(P_x) \leq 1\}$.

Proposition. 3

Let (T_n) be a fuzzy bounded increasing sequence of elements of S, $T_n \leq T_{n+1} \leq M.I$. Then$(T_n)$ F-converges strongly to an element T of s

$$\lim_{n\to\infty} T_n P_x = TP_x(P_x \in P^*(F(H)))$$

Proof.
For m < n,
let $P_{A_{m,n}} = T_n - T_m$.
By the generalized Schwartz inequality with $T = P_{A_{m,n}}$ and $y = P_{A_{m,n}} P_x$,
We have $n(P_{A_{m,n}} P_x)^4 = |(P_{A_{m,n}} P_x, P_{A_{m,n}} P_x)|^2$
$= |P_{A_{m,n}} P_x, P_y|^2$
$\leq (P_{A_{m,n}} P_x, P_x)(P_{A_{m,n}} P_y, P_y)$.
Since $0 \leq P_{A_{m,n}} \leq MI$, we have $(P_{A_{m,n}} P_y, P_y) \leq M^3 n(P_x)^2$
Hence $n(T_n P_x - T_m P_x)^4 \leq M^3 n(P_x)^2 \{T_n P_x, P_x - T_m P_x, P_x\}$.
Since the F-sequence$(T_n P_x, P_x)$is a F-bounded increasing sequence of real numbers, it follows that $(T_n P_x)$ is a F-Cauchy sequencewhich F-converges to an element$TP_x \in P^*(F(H))$.

Proposition. 4
If $T \geq 0, I + T$ is invertible, $(I + T - 1) \geq 0$, and $(I + T) - 1 \in (T)$.
Proof.
Now, We have

$$I \leq I + T \leq (1 + M)I,$$
$$\frac{1}{1+M} \leq P_A \leq I$$

Where $P_A = \frac{1}{1+M}(I + T)$
Therefore $n(I - P_A) \leq n\left(1 - \left(\frac{1}{1+M}\right)I\right) = \frac{M}{1+M} < 1$.

Proposition. 5
If $P_A \geq 0, P_B \geq 0, P_A P_B = P_B P_A$ then $P_A P_B \geq 0$.
Proof.
Since $P_A \in (P_B)$, we have $(P_B)^{\frac{1}{2}} \in (P_A)$ and so

$$P_A P_B = P_A (P_B)^{\frac{1}{2}} (P_B)^{\frac{1}{2}}$$
$$P_A P_B = (P_B)^{\frac{1}{2}} P_A (P_B)^{\frac{1}{2}}$$

Therefore $\left((P_B)^{\frac{1}{2}} P_A (P_B)^{\frac{1}{2}} x, x\right) = (P_A (P_B)^{\frac{1}{2}} x, (P_B)^{\frac{1}{2}} x) \geq 0$.

Theorem.1:
Let $P_A \geq 0$, and let $P_B = 2(P_A)^2 (I + (P_A)^2)^{-1}$. Then
1. $P_B \in (P_A)$,
2. $0 \leq P_B \leq P_A$,
3. $I - P_B = (I - P_A)(I + P_A)(I + (P_A)^2)^{-1}$,
4. if $P_\wp$ is a projective permutable with P_A and $P_\wp \leq P_A$, then $P_\wp \leq P_B$, for some $P_A, P_B, P_\wp \in P^*(F(H))$

Proof.
Proposition (h) implies (1).
That $P_B \geq 0$ is clear since $(P_A)^2$ and $(I + (P_A)^2)^{-1}$ are permutable.
Also $(I + (P_A)^2)(P_A - P_B) = P_A + (P_A)^3 - 2(P_A)^2 = P_A(I - P_A)^2 \geq 0$,

$$P_A - P_B = (I + (P_A)^2) - 1(I + (P_A)^2)(P_A - P_B) \geq 0$$

This proves (2),and (iii) is straight forward.
Let $P_\wp$ be a projection such that $P_\wp \in P_A{}'$ and $P_\wp \leq P_A$. We have

$$P_\wp = (P_\wp)^2 \leq P_\wp P_A \leq (P_A)^2$$
$$P_\wp = (P_\wp)^2 \leq (P_A)^2 P_\wp.$$
$$(I + (P_A)^2)(P_B - P_\wp) = 2(P_A)^2 - (I + (P_A)^2)P \geq 2(P_A)^2 - 2(P_A)^2 P$$
$$= 2(P_A)^2 (I - P_\wp) \geq 0$$

Since $(I + (P_A)^2)^{-1}$ is permutable with all the fuzzy operators concerned,
for some $P_A, P_B, P_\wp \in P^*(F(H))$

$$P_B - P_\wp \geq (I + (P_A)^2)^{-1} 2(P_A)^2 (I - P_\wp) \geq 0.$$

Theorem.2:
Let $P_A \in P^*(F(H))$ be a positive operator, and let the sequence (P_{A_m}) be defined inductively by

$$P_{A_1} = P_A, P_{A_{m+1}} = 2(P_{A_m})^2 (I + (P_{A_m})^2)^{-1} (m = 1,2,\dots)$$

Then

1. $0 \leq P_{A_{m+1}} \leq P_{A_m} (m = 1,2, \ldots)$,
2. The sequence (P_{A_m}) converges strongly to a projection Q belonging to(P_A).
3. $Q \leq P_A$,
4. $(I - P_A)(I - Q) \geq 0$,
5. Q is maximal in the sense if $P_{\wp}$ is a projection permutable e with P_A And satisfying $P_{\wp} \leq P_A$, then $P_{\wp} \leq Q$.

Proof.

(1) This follows at once from theorem 1. (2) and (3). It follows from (1) and Proposition (3) that (P_{A_m}) converges strongly to a positive operator Q with $Q \leq P_A$, and that $Q \in (P_A)$. It remains to p that Q is a projection.

Since $0 \leq P_{A_m} \leq P_A$, we have

$$n(P_{A_m}) \leq n(P_A)(m = 1,2, \ldots);$$

There fore $\lim_{n\to\infty} P_{A_m} P_x = QP_x$ for some $P_A, P_x \in P^*(F(H))$

$\lim_{n\to\infty} (P_{A_m})^2 P_x = Q^2 P_x$ for some $P_A, P_x \in P^*(F(H))$

$\lim_{n\to\infty} P_{A_{m+1}} \{(I + Q^2)P_x - (I + (P_{A_m})^2)P_x\} = 0$, for some $P_x \in P^*(F(H))$

Therefore $(Q - Q^2)^2 = 0$, But $Q - Q^2$ is symmetric, this gives $(\mathrm{Q} - \mathrm{Q})^2 = 0$, hence Q is a projection.

(4) $(\mathrm{I} - \mathrm{P}_{\mathrm{A_m}}) = (\mathrm{I} - \mathrm{P}_{\mathrm{A_{m-1}}})(\mathrm{I} + \mathrm{P}_{\mathrm{A_{m-1}}})(\mathrm{I} + (\mathrm{P}_{\mathrm{A_{m-1}}})^{-1})$

$$(\mathrm{I} - \mathrm{P_A})(\mathrm{I} - \mathrm{P}_{\mathrm{A_m}}) \geq 0$$

$$(\mathrm{I} - \mathrm{P_A})(\mathrm{I} - \mathrm{Q}) \geq 0$$

(5) Let $\mathrm{P}_{\wp}$ be a projection permutable with $\mathrm{P_A}$ such that $\mathrm{P}_{\wp} \leq \mathrm{P_A}$.

By repeated application of theorem1 and (4),$\mathrm{P}_{\wp}$ is permutable with $\mathrm{P}_{\mathrm{A_m}}$

And $\mathrm{P}_{\wp} \leq \mathrm{P}_{\mathrm{A_m}}$. In the limit we have $\mathrm{P}_{\wp} \leq \mathrm{Q}$.

CONCLUSION

The adjoint Fuzzy linear operator's definitions are insert able in this work into a fuzzy Hilbert space. In order to demonstrate several fundamental theorems and elementary propositions, adjoint and self-adjoint fuzzy operators in FH-space were used.

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