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CAUCHY SEQUENCE IN FUZZY BI-NORMED LINEAR SPACES

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ABSTRACT
In this paper we introduce the notion of fuzzy point and Cauchy sequence in fuzzy bi-normed linear space and some basic results related to these notions. Certain results in connection with the a fuzzy bi-normed convergence and Cauchy sequence are also analyzed

Keywords: Fuzzy point, Bi-norm, Convergence, Cauchy Sequence, Complete.

1. INTRODUCTION
The concept of fuzzy set was introduced by Zadeh [7] in 1965. A. White [6] introduced the 2-Banach Space. A.K. Katsaras [5] introduced the Fuzzy topological vector spaces. Meenakshi A.R. and Cokilavany [3] introduced the concept of fuzzy 2-normed linear spaces.Nagoorgani A. and Kalyani G. [4] introduced the Bi-normed sequences in fuzzy matrices. [2] Introduced the On fuzzy 2-metric space, Cokilavany.D.The organization of the paper is as follows: Section 2, provides some preliminary results which are used in this paper. In section 3, we have discussed convergence and F- Cauchy sequence theorem is proved in the realm of fuzzy bi-normed linear spaces in$P^*(X)$.

2 PRELIMINARIES
Let X be a nonempty set, $I = [0,1]$ be the unit interval. The pair $(x, \alpha), x \in X, \alpha \in I$ is called a fuzzy point in X, denoted by P_x^α or $P(x, \alpha)$, some times simply P Fuzzy point $P^{1-\alpha}$ is called the dual point of P_x^α usually the dual point of P is denoted by P^*. The set of all fuzzy points in X is denoted by $P^*(X) = \{P_x^\alpha : x \in X, \alpha \in [0,1]\}$. For any fuzzy set $A \in I^x$, the collection of all mappings of Xinto I. we say that $P_x^\alpha \in A \Leftrightarrow \alpha < A(x)$ or $A(x) = 1$ and $P_x^\alpha \in A \Leftrightarrow \alpha \leq A(x)$and $A(x) \neq 0$.Let $P_x^\alpha, P_y^\beta \in P^*(X)$ be fuzzy points, then we say that $P_x^\alpha \leq P_y^\beta \Leftrightarrow x = y, \alpha \leq \beta$ and $P_x^\alpha < P_y^\beta \Leftrightarrow x = y, \alpha < \beta$.

Definition 2.1
A fuzzy bi-norm for a real or complex linear space X is a real valued function θ defined on $P^*(X) \times P^*(X)$ to [0, 1] satisfying the following conditions. For any $P_1, P_2, P_3 \in P^*(X)$ and $\alpha \in [0,1]$

(i) $\theta(P_1, P_2) = 0$ if and only if $P_1 \leq P_2$ or $P_2 \leq P_1$

(ii) $\theta(P_1, P_2) = \theta({P_2}^*, {P_1}^*)$

(iii) $\theta(rP_1, P_2) = |r|\theta(P_1, P_2), \forall r \in [0,1]$

(iv) $\theta(P_1 + P_2, P_3) \leq \theta(P_1, P_3) + \theta(P_2, P_3)$

Definition 2.2:
Let A be in $P^*(X)$ and α be in [0,1] such that $\|A\| = \alpha$. Then the pair (A,α) is called a fuzzy point in M_n(F) and it is denoted by P_A^α. The dual fuzzy point for P_A^α is the point with

α -norm (1-α) denoted by $P^* = P^{1-\alpha}$

Definition 2.3:
The set of all fuzzy point in M_n(F) is given by $P^*(X) = \{P_A^\alpha / A \in P^*(X), \alpha \in [0,1]\}$

In F we follow the usual ≤ order relation correspondingly we define an order relation in

$P^*(X)$as follows.

Definition 2.4:
We define $P_A^\alpha < P_B^\beta$ if and only if α <β and $P_A^\alpha = P_B^\beta$ if and only if A = B

(then automatically α = β)

Definition 2.5:
A sequence$\{P_{A_n}\}$ in this fuzzy binormed linear space $\{M_n(\text{F}), \theta\}$ is called a sequence with respect to bi-norm if there exists P_A, P_B in $P^*(M_n(\text{F}))$ such that $P_A \neq P_B$,

$\lim \theta(P_{A_n} - P_{A_m}, P_A) = 0and\lim \theta(P_{A_n} - P_{A_m}, P_B) = 0$

Definition 2.6:
A sequence$\{P_{A_n}\}$ in $\{M_n(\text{F}), \theta\}$ is called a convergent sequence if there is a P_A in $P^*(M_n(\text{F}))$ such that

$\lim \theta(P_{A_n} - P_A, P_B) = 0$for all P_B in $P^*(M_n(\text{F}))$

3. Convergence and F- Cauchy Sequence

Theorem 3.1: Let (X, θ) be a fuzzy Bi-Normed linear space and let $\{P_{x_n}\}$ and $\{P_{y_n}\}$ be two sequence in $P^*(\text{X})$ such that $P_{x_n} \to P_x$and $P_{y_n} \to P_y$, where P_x , $P_y \; in \; P^*(\text{X})$**,** (i) $P_{x_n} + P_{y_n} \to P_x + P_y$, (ii) If $Lim(X, \theta) \geq 2$, then$P_{x_n}P_{y_n} \to P_xP_y$, (iii) $kP_{x_n} \to k\, P_x$ for k ϵ [0,1]

Proof:
(i) Suppose that $P_{x_n} \to P_x$and $P_{y_n} \to P_y$, For each ϵ >0 , P_z ϵ $P^*(\text{X})$ and k ϵ [0,1]

$\theta((P_{x_n} + P_{y_n}) - (P_x + P_y), P_z) \leq \theta(P_{x_n} - P_x, P_z) + \theta(P_{y_n} - P_y, P_z) = \theta((P_{x_n} - P_{y_n}) - (P_x - P_y), P_z) \geq \frac{\varepsilon}{2}$,= $\theta((P_{x_n} - P_x), P_z) \geq \frac{\varepsilon}{2}$ and $\theta((P_{y_n} - P_y), P_z) \geq \frac{\varepsilon}{2}$, $\theta((P_{x_n} + P_{y_n}) - (P_x + P_y), P_z)$ =lim $\theta((P_{x_n} - P_x), P_z) = 0$ for all $P_z \epsilon\, P^*(\text{X})$ and lim $\theta((P_{y_n} - P_y), P_z) = 0$ for all $P_z \epsilon\, P^*(\text{X})$. Therefore lim$\theta((P_{x_n} + P_{y_n}) - (P_x + P_y), P_z) = 0$ for all $P_x, P_y, P_z \;\epsilon\; P^*(\text{X})$.

(ii) Since $P_{x_n} \to P_x$,$\theta((P_{x_n} - P_x), P_z) < 1$ in $P^*(\text{X})$. $\theta(P_{x_n}P_{y_n} - P_xP_y, P_z) \leq \theta(P_{x_n}, P_z)\, \theta(P_{y_n} - P_y, P_z) + \theta(P_y, P_z)\, \theta(P_{x_n} - P_x, P_z)$, We have $\theta(P_{x_n}, P_z) \leq \theta(P_x) + 1$

$\theta(P_{x_n}P_{y_n} - P_xP_y, P_z) \leq (\theta(P_x) + 1)\, \theta(P_{y_n} - P_y, P_k)\, \theta(P_y, P_k)\, \theta(P_{x_n} - P_x, P_k)$ …..(1)

Let Ɛ> 0 and , $P_z \epsilon P^*(\text{X})$, Choose η >0 such that $0 < 2\eta < \frac{\varepsilon}{\theta(P_x)+\theta(P_y)+1}$ ………(2)

Since $P_{x_n} \to P_x$, and $P_{y_n} \to P_y$,$\theta(P_{x_n} - P_x, P_z) < \eta$ and $\theta((P_{y_n} - P_y), P_z) < \eta$ in $P^*(\text{X})$

$\theta(P_{x_n} - P_x, P_k) \cap \theta(P_{x_n} - P_x, P_z) \cap \theta(P_{y_n} - P_y, P_z)$ in $P^*(\text{X})$ and for each $P_x, P_y, P_z \;\epsilon\; P^*(\text{X})$.

From (1) and (2) ,$\theta(P_{x_n}P_{y_n} - P_xP_y, P_z) < \varepsilon$. Therefore $\theta(P_{x_n}P_{y_n} - P_xP_y, P_z) \geq \varepsilon$ in $P^*(\text{X})$. Hence $P_{x_n}P_{y_n} \to P_xP_y$.

(iii) Since $P_{x_n} \to P_x$, $\theta(P_{x_n} - P_x, P_z) \geq \varepsilon$. Now let C≠0 , Ɛ > 0 and $P_x, P_z \;\epsilon\; P^*(\text{X})$ such that θ(k P_x, P_z) = $|k|\theta(P_x, P_y)$, let $\theta(k\, P_{x_n} - kP_x, P_z) \geq \varepsilon$, Now prove that $\varepsilon \subset \varepsilon_1$, Let m Є $\theta(k\, P_{x_n} - kP_x, P_z)$. For some $\varepsilon_1 > 0, \varepsilon \leq \theta(k\, P_{x_m} - kP_x, P_z) \leq$ |k| $\theta(k\, P_{x_m} - kP_x, P_z)$

$\frac{\varepsilon}{|k|} \leq \theta(k\, P_{x_m} - kP_x, P_z)$,$\varepsilon \leq \varepsilon_1$.Therefore m $\epsilon\theta(k\, P_{x_m} - kP_x, P_z)$

Theorem 3.2: Let (X,θ) be a fuzzy Bi-Normed linear space and let $\{P_{x_n}\}$ be a F-Cauchy Sequence. If $\{P_{x_n}\}$ has a subsequence $\{P_{x_{n_k}}\}$ converging to P_x then $P_{x_n} \to P_x$.

Proof :
Let > 0 , $P_{x_n}, P_{x_{n_k}} \in P^*(\text{X})$ and $P_z \in P^*(\text{X})$,$\theta(P_{x_n} - P_{y_n}, P_z) > \frac{\varepsilon}{2}$. Since $\{P_{x_{n_k}}\} \to P_x$, $\theta(P_{x_{n_k}} - P_x, P_z) \geq \frac{\varepsilon}{2}$ for all $n \geq n_0 \in \aleph$, $\theta(P_{x_n} - P_x, P_z) = \theta(P_{x_n} - P_{x_{n_k}} + P_{x_{n_k}} - P_x, P_z) \geq \theta(P_{x_n} - P_{x_{n_k}}, P_z) + \theta(P_{x_{n_k}} - P_x, P_z)$, $\theta(P_{x_n} - P_x, P_z) \geq \varepsilon$. Therefore ,$\{P_{x_n}\}$ has a subsequence $\{P_{x_{n_k}}\}$ converging to P_x in $P^*(\text{X})$.

Theorem3.3: Let (X,θ) be a fuzzy Bi-Normed linear space and let $\{P_{x_n}\}$ be a sequence $P^*(\text{X})$. If $\{P_{x_n}\}$ is a convergent then it is a cauchy sequence.

Proof : Let $\{P_{x_n}\}$ converges to P_x, Lim $\theta(P_{x_{n_k}} - P_x, P_z) = 0$ for each $P_z \epsilon P^*(\text{X})$. There exists $n_0 \in \aleph$ such that $\theta(P_{x_{n_k}} - P_x) > \varepsilon$ for every $\varepsilon > 0$.Each$P_2 \in P^*(x), \theta\left(P_{x_{n_k}} - P_{y_{n_k}}, P_z\right) \leq \theta\left(P_{x_{n_k}} - P_x, P_z\right) + \theta\left(P_{y_{n_k}} - P_x, P_z\right) <$ 2 ∈ for every ∈> 0, each $P_2 \in P^*(x), \theta\left(P_{x_{n_k}} - P_{y_{n_k}}, P_z\right) <$ 2 ∈. Hence $\{P_{x_n}\}$is an F-Cauchy Sequence.

Example3.5: Let (X, θ^1) be a Bi-normed linear space. Define $\theta : X \to [0,1]$by

$$\theta\ (P_x)(t) = \begin{cases} 1\ if\ t\ >\ \theta(P_x) \\ 0\ if\ t\ >\ \theta(P_x) \end{cases}$$

Then $\theta_2(P_x) = \left(\theta(P_1^*(x), \theta(P_2^*(x)\right) \ \forall\ \alpha \in (0,1]$, It is verify that,

i. $\theta(P_1, P_2) = 0\ if\ and\ only\ if\ P_1 \leq P_2\ or\ P_2 \leq P_1$

ii. $\theta(cP_1, P_2) = |c|\ \theta(P_1, P_2)$for all c ∈ K.

iii. $\theta(P_1 + P_2, P_3) \leq \theta(P_1, P_3) + \theta(P_2, P_3)$

Then (X, θ) is called a fuzzy bi-normed linear space with respect to the Bi-norm [$\theta(P_1, P_2)$is defined for $P_1 = P_2$ as 0 by (i); therefore $\theta(P_1, P_2) \neq 0$ is defined for $P_1 < P_2$].

Thus (X, θ) is a fuzzy Bi-normed linear space. Let $\{P_{x_n}\}$ be a Cauchy sequence in $P^*(X)$.

$\lim \theta\left(P_{x_n} - P_{y_n}, P_z\right) = 0 \implies \{P_{x_n}\}$be a Cauchy sequence in $P^*(X)$.

Hence (X, θ) is complete, (X, θ) is a real fuzzy Banach Space.

Theorem3.6: Every fuzzy Bi-norm linear space with normal constant K is complete.

Proof: Let (X, θ) be a fuzzy Bi-normed linear space with normal constant K.Let $\{P_{x_n}\}$ be a Cauchy sequence in $P^*(X)$.Let dim X = n and $\{e_1, e_2, \ldots \ldots \ldots, e_n\}$ be a basic for $P^*(X)$. Then each $\{P_{x_n}\}$has a unique $P_{x_n} = \beta_1^n e_1 + \beta_2^n e_2 + \cdots \ldots \ldots + \beta_m^n e_m$. Since $\{x_n\}$ is a Cauchy sequence for every $e \in P^*(X)$ with $\theta(e) \gg 0$ there exists a positive integer N such that $\theta\left(P_{x_n} - P_{y_n}\right) \ll \theta(e)$.There exists $c \in P^*(X)$ with $\theta(c) > 0$ such that $\theta(\varepsilon) \gg \theta\left(P_{x_n} - P_{x_k}, h\right) \geq \theta(c) \sum_{i=1}^m \left|P_{\beta_i^{(n)}} - P_{\beta_i^{(k)}}, P_\beta\right|$,$\theta(c) \sum_{i=1}^m \left|P_{\beta_i^{(n)}} - P_{\beta_i^{(k)}}, P_\beta\right| \leq \theta(e) \leq k\ \theta(e)\ \forall\ n, k \geq N$, $\theta(c) \sum_{i=1}^m \left|P_{\beta_i^{(n)}} - P_{\beta_i^{(k)}}, P_\beta\right| \leq k\ \theta_\alpha(e) \implies \theta_\alpha(c) \left|P_{\beta_i^{(n)}} - P_{\beta_i^{(k)}}, P_\beta\right| \leq k\ \theta_\alpha(e)$. Since $c \in P^*(X)$is arbitrary of the m sequence $\left\{\beta_i^{(n)}\right\}$ is Cauchy in R.Since R is complete, thus each $\beta_i^{(n)}$ converges and denote by β_i are their limits for each $i = 1,2, \ldots \ldots, m$.We define $P_x = P_{\beta_1}\varepsilon_1 + P_{\beta_2}\varepsilon_2 + \cdots \ldots \ldots + P_{\beta_m}\varepsilon_m$, $\theta\big(P_{x_n} - P_x, P_z\big) = \theta(\sum_{i=1}^m \left|P_{\beta_i^{(n)}} - P_{\beta_i^{(k)}}, P_\beta\right| e_i \leq \left|P_{\beta_1^{(n)}} - P_{\beta_1}\right| \theta(e_1) + \left|P_{\beta_2^{(n)}} - P_{\beta_2}\right| \theta(e_2) + \cdots \ldots \ldots + \left|P_{\beta_m^{(n)}} - P_{\beta_m}\right| \theta(e_m)$.Since $\left|P_{\beta_i^{(n)}} - P_{\beta_i}\right| \to 0\ as\ n \to \infty$.Therefore $\theta\big(P_{x_n} - P_x, P_z\big) \to 0\ as\ n \to \infty$.Thus the Cauchy sequence $\{x_n\}$converges to $x \in X$.Hence, $\{P_{x_n}\}$is arbitrary it follows that X is complete.

CONCLUSION

In this paper, we consider the fuzzy Bi-norm and some basic concepts. Cauchy sequence of fuzzy Bi-normed linear space in $P^*(X)$ is convergent, Since these theorem have many applications in functional analysis. Numerical examples are given to clarify the developed theory and the proposed fuzzy Bi-normed linear space.

REFERENCES

[1] T.Bag. Some Fundamental Theorem in Felbins Type Fuzzy Normed Linear Space. IJMSC, Vol. 1. No, 2, 2011.

[2] Cokilavany. D., On fuzzy 2-metric space,International journal of Journal of fuzzy Mathematic Archives, volume 8(No.2) 2015 (139-145)

[3] Meenakshi A.R. and Cokilavany D., On fuzzy 2-normed linear spaces, The Journal of fuzzy mathematics, volume 9(No.2) 2001 (345-351)

[4] Nagoorgani A. and Kalyani G. Binormed sequences in fuzzy matrices. Bulletin of Pure and Applied Science. Vol. 22E(No.2) 2003; P. 445-451

[5] A.K. Katsaras, Fuzzy topological vector spaces II, Fuzzy sets and System 12(1984),143-154

[6] A. White, 2-Banach Space, Math. Nachr 42(1969),43-60

[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.

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