MATHEMATICAL MODELLING: MARKOV DECISION PROCESS IN STOCHASTIC RESOURCE MANAGEMENT SYSTEMS

A SYNOPSIS
submitted
by

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for the award of the degree
of

DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

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DECEMBER 2018

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SYNOPSIS

1. Introduction

Markov decision theory provides powerful tool for analysis of the probabilistic sequential decision processes. This model is an outgrowth of the Markov process and Dynamic Programming.

The concept of Dynamic Programming(DP) being developed by Bellman in the early 1950s, is a computational approach for analyzing sequential decision processes with a finite planning horizon. The basic idea involved in DP are states, the principle of optimality and functional equations.

(1) **States**

This is a set of positions which are possessed by the system at various time points.

(2) **Principle of optimality**

Future decisions for all future stages constitute an optimal policy regardless of the policy adopted in all the preceding stages.

(3) **Functional Equations**

The recurrence equations formed with the state variables, forward recursive equation or backward recursive equation are called **functional equations**

Markov decision model has many potential applications in inventory control, maintenance, manufacturing, telecommunication systems and computer science among others.

2. Literature review

An implicit assumption in most inventory models is that inventory is depleted at a rate equal to the demand rate. In spite of this model is realistic for production/manufacturing industries, it is less realistic for service facility. If inventory is depleted according to the demand rate then the queues are negligible, otherwise it is depleted according to the service rate and customers form a queue for service. For example, in the hospitals where units of blood are necessary for surgery, installing software's in a computer shop, packing selected gift in shop, fitting bumper for a car in service station and serving foods at restaurants, these kind of service facility models are used.

In all works reported in inventory prior to 1993 it was assumed that the time required to serve the item to the customer is negligible. Berman, Kaplan and Shimshak [3] is the first attempt to introduce positive service time in inventory, where it was assumed that service time is deterministic. A logically related model was studied by He et al. [6], which analysed a Markovian inventory - production system, where customer's demands arrive at a workshop and are processed by a single machine with random batch size. Latter Berman and Kim [4] extended these results to random service time. Some other work reported in inventory with service time are Berman and Sapna [5] investigate inventory control at a service facility, which uses one item of inventory each service provided. Assuming the Poisson arrival process, arbitrarily distributed service times and zero lead time they analyse the system with the restriction of finite waiting space is finite. Under a specific cost structure they derived the optimum ordering quantity that minimizes the long run expected cost rate.

Arivarignan and Elango et al. [1] considered a Markovian inventory system in which the size of the space for the waiting customers is assumed to be infinite with Poisson arrivals and exponentially distributed leadtimes and service times. Arivarignan and Sivakumar [2] considered an inventory system with arbitrary distribution for inter-occurrence of demands, exponential service time and exponential lead time. Sivakumar and Arivarignan [14, 15] emphasized an elaborate work on stochastic inventory systems with service facilities.

Yadavalli et al. [16] studied an inventory system in which demand points of customers from a renewal process and the service times are to be distributed as negative exponential. For a complete survey of Inventory systems with service facility see Krishnamoorthy et al. [8]. Recent papers in an inventory system with service facility are studied in depth by Satheesh kumar and Elango [11, 12], Krishnamoorthy and Narayanan [10], Yadavalli et al. [17], Shophia Lawrence et al. [13], Krishnamoorthy et al. [9], Hamadi et al. [7].

In this thesis dynamic control at service facility system is considered for indepth study. Only optimal parameters are obtained in a specific cost structure. In this thesis we studied a discrete time MDP model to analyze stochastic service facility system. Together with this system many other discrete MDP problems are solved using LPP or Value-iteration or Policy-iteration methods.

3. Organization of the Thesis

Scare resources like service facility and inventory should be maintained optimally to serve the customers efficiently.

In thesis, we presents different analytic models for discrete/ continuous time Markov Decision Processes dealing with service facility system with/ without inventory for service completions.

The thesis entitled ***Mathematical Modelling: Markov Decision Process in Stochastic Resource Management Systems*** is divided into 4 parts.

In part I (Chapter 1,2), a short introduction to queueing theory, inventory control and stochastic process theory is presented. The relevant materials on variety of inventory control policies and the description of service facility with inventory management are also provided. When analyzing these models we exploit the discrete time queue structure to fit the model with real life situations. In the introduction part, we also present some terminology that is used throughout the thesis and also few fundamental results on the subject under study without going into much detail.

In Part II (Chapter 3,4,5), we present some discrete time MDP models.

In Chapter 3, we present a discrete time MDP model for a service facility system with inventory for service completion. Optimal admission policy is obtained by minimizing average cost using Policy iteration method in MDP formulations.

In Chapter 4, a MDP problem in which both admission and service rate control has been studied in detail. Here, we use policy iteration method to optimize the expected total reward. Numerical example is provided to illustrate the model.

In chapter 5, an indepth study of admission control in a discrete time service facility with inventory with customers differentiation and MDP formulation is done by controlling the arrivals according to the cost parameter.

In Part III (Chapter 6, 7), some discrete time inventory control in service facility system is studied.

In chapter 6, two model has been formulated under MDP frame to control the inventory reordering process in a discrete time service facility. In the first model Discrete time service facility system with modified ordering policy is considered for study with LPP method. In model 2, Linear programming method is used to find the optimal inventory control for discrete time service facility system with (s, S) ordering policy. We find an optimal ordering policy by using average cost criteria under Linear programming technique.

In chapter 7, we present a semi - MDP model for a service facility system with vacation for server. Optimal ordering policy is obtained by minimizing average cost using Linear programming method in MDP formulations.

Finally, in Part IV, Chapter 8, we present conclusion and further direction for work related to this thesis.

4. Research Methodology

Most of the models in MDP problems are taken up for indepth study using the tools:

(i) Policy-iteration method

(ii) LPP method and

(iii) Value-iteration method

(i) Policy-Iteration

The policy-iteration algorithm converges after finite number of iterations to an average cost optimal policy. The policy-iteration algorithm is empirically found to be a remarkably robust algorithm that converges very fast in specific problems.

(ii) Linear Programming Approach

In this thesis, MDP problems are taken up for indepth study using LPP method. Consider the objective function $f(x)$ as maximization of profit or minimization of cost, subject to the constraints involving all decision variables and quantity of resources will give the LPP formulation. Solving the problem by the standard Simplex method, We get optimal solution for the MDP problem.

(iii) Value-Iteration

The value-iteration algorithm computes recursively a sequence of value functions approximating the minimal average cost per time unit. The value functions provide lower and upper bounds on the minimal average cost and under a certain aperiodicity condition these bounds converge to the minimal average cost.

5. Experiments and Results

Model 1: Admission Control in a Discrete time Service Facility with Inventory

Model Description:

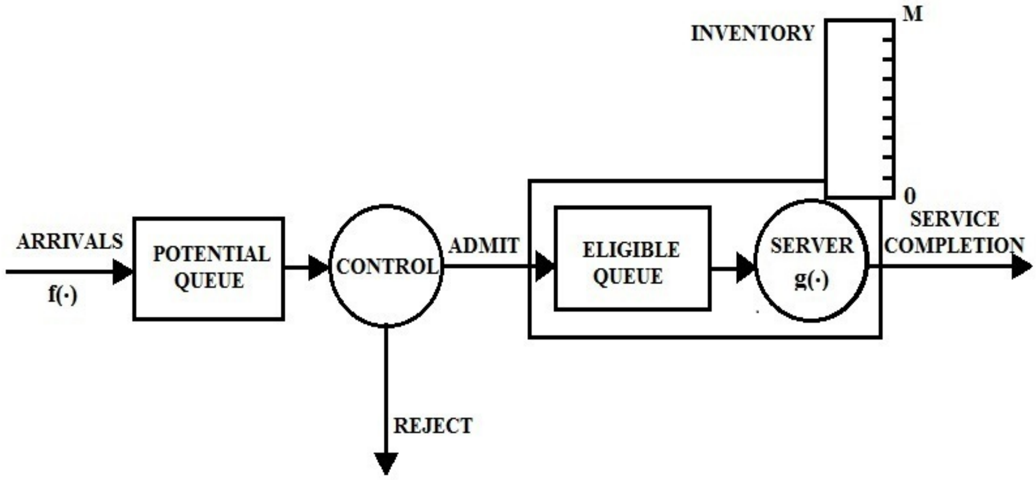

Figure 1: The MDP model for service facility system with inventory

(i) The system is observed every $\eta > 0$ unit of time and the decision epochs are $0,\ \eta,\ 2\eta,\ \ldots$,(infinite planning horizon).

(ii) Admissions to the service facility is controlled, by observing the number of customers in the potential queue.

(iii) Arriving number of customers to service facility system follows a probability distribution $f(\cdot)$ and the arriving customers are placed in potential customers queue. Possible number of service completion follows a general probability distribution $g(\cdot)$.

(iv) Only the eligible queue(main queue) customers get service.

(v) No partial service completion allowed during any period.

(vi) All serviced customers take unit item from inventory and depart the system at the end of period.

(vii) The $(0, M)$ policy is adopted with instantaneous replenishment. An order for M items is placed when inventory level reaches zero at the decision epoch.

(viii) Decision to order additional stock is made at the beginning of each period and delivery occurs instantaneously.

MDP Formulation:

We consider the problem on MDP having five components(tuples) $(T, S, A_s, p_t(\cdot|\,), c_t(\,))$

Decision Epochs:

$$T = \{0, \eta, 2\eta, \dots\}$$

State Space:

$$\begin{aligned} S &= S_1 \times S_2 \times S_3 \\ &= \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots, M\}, \end{aligned}$$

$$\begin{aligned} S = \{(j_1, j_2, j_3) : \; & j_1 \text{ denotes the number of customers in the system,} \\ & j_2 \text{ denotes the number of customers in potential queue} \\ & j_3 \text{ denotes the number of items in inventory}\} \end{aligned}$$

Actions:

Number of customers admitted is the decision variable, then the action set at (j_1, j_2, j_3) is:

$$A_{(j_1,j_2,j_3)} = \{0, 1, \dots, j_2\}, \quad A = \bigcup_{s \in S} A_s$$

Transition Probability:

$$p_t(s'|s,a) = \begin{cases} g(j_1 + a - j_1')\ f(j_2') & \text{if} \quad a + j_1 > j_1' > 0 \\ \sum\limits_{i=j_1+a}^{\infty} g(i)\ f(j_2') & \text{if} \quad j_1' = 0,\ a + j_1 > 0 \\ f(j_2') & \text{if} \quad j_1' = a + j_1 = 0 \\ 0 & \text{if} \quad j_1' > a + j_1 \geq 0 \\ g(j_3 - j_3')\ f(j_2') & \text{if} \quad j_3 > j_3' > 0 \\ \sum\limits_{i=1}^{M} g(i)\ f(j_2') & \text{if} \quad j_3' = 0,\ j_3 > 0 \\ 0 & \text{if} \quad j_3' > j_3 > 0 \end{cases}$$

where $s = (j_1, j_2, j_3),\ s' = (j_1', j_2', j_3'),\ a \in A_{(j_1,j_2,j_3)}$.

Reward/ cost structure:

The cost occurred during the period t is given by

$$c_t(s,a) = C \times E[\min\{\min(Y_t, j_1 + a), j_3\}] \ + \ w(j_1 + a) \ + \ h(y)$$

$$a \in A = \bigcup_{s \in S} A_s, \quad s = (j_1, j_2, j_3)$$

The expected number of service completion in period t is

$$E[\min\{\min(Y_t, j_1 + a), j_3\}] = \sum_{i=1}^{j_1+a-1} ig(i) + (j_1 + a) \sum_{i=j_1+a}^{\infty} g(i) + \sum_{i=1}^{M} i\ g(i)$$

The stationary cost structure consist of three components: a service cost C per completed service and a waiting cost $w(x)$ per period when there are x customers in system(eligible queue) and a holding cost $h(y)$ per period when there are y items in inventory.

Model 2: Admission Control in a Discrete time MDP with Inventory with Customers Differentiation

Model Description:

(i) The system is observed every $\eta > 0$ unit of time and the decision epochs are $0,\ \eta,\ 2\eta,\ \ldots,$ (infinite planning horizon).

(ii) Admissions to the service facility is controlled, by splitting the queue into Eligible queue and Potential queue. Potential queue has two types of customer called priority ($T^{(1)}$) and non-priority ($T^{(2)}$) customers ($T^{(1)}$ and $T^{(2)}$ used as symbol).

(iii) At each decision epoch the controller observes the number of items in stock, number of customers in the system (Eligible queue + Server) and the number of customers (classified) in the potential queue.

(iv) Capacity of both eligible and potential queues is finite say $N_1,\ N_2 \leq \infty$ respectively. Assume that the maximum capacity of inventory is M (finite).

(v) Arriving customers to service facility system follows a probability distribution $g_1(\cdot)$ and $g_2(\cdot)$ for priority and non-priority customers respectively and the arriving customers are placed in potential queue. Possible service completions in each period follows a probability distribution $f(\cdot)$.

(vi) No partial service completion allowed during any period.

(vii) All serviced customers depart the system at the end of period.

(viii) The $(0, M)$ policy is adopted with instantaneous replenishment. An order for M items is placed when inventory level reaches zero at the decision epoch.

(ix) Decision to order additional stock is made at the beginning of each period and delivery occurs instantaneously.

Model 3: Admission and Service Control in a Discrete time Service Facility

Model Description:

1. The system is observed every $\eta > 0$ unit of time and the decision epochs are $0, \eta, 2\eta, \ldots, L\eta,\ L \leq \infty$.

2. Admissions to the service facility is controlled, by observing the number of customers in the potential queue.

3. Service is controlled by selecting a service rate (parameter of the probability distribution) from the set of rates
$B = \{b_\beta : \beta = 0, 1, ..., k\}$. The service rates can be changed depending on the number of customers in the system.

4. Arriving number of customers to service facility system follows a probability mass function $p(\cdot)$ and the arriving customers are placed in *potential queue*.

5. Possible number of service completion follows a general probability mass function $q_{b_\beta}(\cdot)$ with rate $b_\beta (\in B)$. The controller uses $q_{b_\beta}(\cdot)$ in period t and uses $q_{b'_\beta}(\cdot)$ in period $t+1$, $b_\beta \neq b'_\beta$ means that a sever change occurs.

6. All serviced customers depart the system at the end of period.

Model 4: Inventory Control in Discrete Time Service Facility (MDP) with Modified Ordering Policy

Model Description:

Consider a discrete time service facility system in which inventory is maintained to serve the customers. The infinite time horizon considered is divided into intervals of equal length say η. We assume that the system activities like arrival, service and departure occurs only at the points of the time intervals (slots) with $\eta = 1$. That is the events are deemed to occur at a same time. The demanded items from the system is issued to the customer only after some random service time. In general retail stores, will not or cannot be allowed to wait for long period of time. In our problem the customer (or machine/car) has to wait until the spare part (inventory) is taken from stock to complete the service. Also we assume that only one item from inventory is used per customer.

- The maximum capacity of the inventory is assumed to be S and the waiting space size is finite, say N.

- A customer who find the waiting space full (N customers) is forced to balk the system.
- Customers arrive to the service facility system following Bernoulli process with probability p and $\bar{p}$ denote the non-arrival of customers and are served according to FCFS queue discipline.
- If the server is free and the inventory level is positive, the arriving customer immediately enter the server, otherwise join the queue (waiting space).
- The service times for customers are independent and geometrically distributed with probability q, where $\bar{q} = 1 - q$ is the probability that a customer does not complete his service.
- One item from inventory is used for a customer to finish the service. Each customer leaves the system with one item at the end of the period only.
- An order decision for replenishment is placed whenever the inventory level is $\leq s$ (prefixed level) as per decision at the time slot. The size of the order is adjusted at the time of replenishment, so that immediately after a replenishment, the inventory level becomes S.
- The order is delivered after lead time having geometric distribution with parameter $r > 0$, where $\bar{r} = 1 - r$ is the probability that ordered items not received.

Discrete time systems are concerned with the happenings of events within the period, that is with in the time epochs t and $t + 1, t = 0, 1, 2, \ldots$. We assume that first demand, then service completion and finally the replenishment event occurs, in a period.

Model 5: Optimal Inventory Control in a Service Facility System with Vacation for Server

Model Description:

Consider a continuous review service facility inventory system with a maximum inventory of S units, a single server and a finite waiting spaces say N. Thus a customer who sees N customers ahead leaves the system without getting service. One unit from inventory is used up to serve one customer. The following assumptions are made:

- The arrival of customers form a Poisson process with rate $\lambda > 0$.
- An order for $Q(= S - s > s)$ items is placed whenever the inventory level drops to a prefixed level, say s, and the items are received only after a random time, which is distributed as exponential with parameter $\gamma(> 0)$.
- If the stock is depleted to zero or no customer is in the system, the server goes on a vacation, for random time period which has exponential distribution with parameter $\delta(> 0)$. At the end of a vacation period, if the stock is still empty or customer level is zero, the server takes another vacation, else he starts service.
- The service times follow an exponential distribution with parameter $\mu(> 0)$.

6. Conclusion

At present the discrete stochastic modelling has covered wide area of applications: queueing, inventory, reliability and retrial queues. Communication and computer systems engineering applications are plenty. Throughout the thesis, we concentrated only on discrete time models for service facility system. We tried to model MDP, by controlling customer's admission, service rate and also the inventory replenishment. In future are can do exhaustive research in this area became most of the model problem we studied are open nature. Stochastic lead time, reneging, customers differentiations and perishing products are all need indepth study.

List of Publications of Ms. P. Maheswari

1. *Discrete MDP Problem with Admission and Inventory Control in Service Facility Systems*, (with Selvakumar, C. and Elango, C.), International Journal of Computational and Applied Mathematics, ISSN 1819 - 4966, volume 12, Number 1, pp 478 - 493, (2017).

2. *Discrete time MDP with Inventory Maintained to Satisfy Two Types of Cusomers*, (with Selvakumar, C. and Elango, C.), Proceedings of the National Conference on Recent Advancements in Pure and Applied Mathematics, July 26 - 27, (2017).

3. *Optimal Admission and Service Control in a Discrete time Service Facility Systems: MDP Approach*, (with Elango, C.), Annals of Pure and Applied Mathematics, ISSN: 2279 - 087X (P), 2279 - 0888 (online), volume 15, No. 2, 315 - 326, (2017).

4. *Optimal Control of Customers to the Service Facility with two Types of Customers*, (with Selvakumar, C. and Elango, C.), International Journal of Fuzzy Mathematical Archive, ISSN: 2320 - 3242 (P), 2320 - 3250 (online), volume 15, No. 2, 217 - 226, (2018).

5. *Discrete MDP Problem in Service Facility Systems with Inventory Management*, (with Selvakumar, C. and Elango, C.), International Journal of Computational Systems Engineering (accepted).

List of Papers Presented at Conference

1. *Discrete MDP Problem in Service Facility Systems with Inventory Management*, **ICMMCMSE- 2017**, organized by Alagappa University, Karaikudi, February 20 - 22, (2017).

2. *Discrete MDP problem with Admission and Inventory Control in Service Facility Systems*, **National Conference on Mathematical Modelling** organized by Cardamom Planters' Association College, Cardamom Planters' Association College, Bodinayakanur, March 30 - 31, (2017).

3. *Discrete time MDP with Inventory Maintained to Satisfy Two Types of Cusomers*, **National Conference on Recent Advancements in Pure and Applied Mathematics**, organized by Nadar Saraswathi College of Arts and Science, Theni, July 26 - 27, (2017).

4. *Optimal Admission and Service Control in a Discrete time Service Facility Systems: MDP Approach*, **ICOMAC 2017** organized by Jamal Mohamed College, Tiruchirappalli, December 11, (2017).

5. *Optimal Control of Customers to the Service Facility with two Types of Customers*, **ISOAAM 2018**, organized by Hajee Karutha Rowther Howdia College, Uthamapalayam, January 11, (2018).

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